S.-T. Yau College Student Mathematics Contests 2022

Analysis and differential equations overall

Problem 1. Let $f : \mathbf{C} \to \mathbf{C}$ be a non-constant holomorphic function.

1) Prove that the image of f is dense in $\mathbf{C}$.

2) Prove that the image of f can miss only one point in $\mathbf{C}$. (Hint: The universal cover of $\mathbf{C} - \{0, 1\}$ is the unit disk.)

Problem 2. Let K be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Define

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \mathrm{d}y.$$

(1) Suppose that $K \in L_x^\infty L_y^1 \cap L_y^\infty L_x^1$. Show that T is a bounded operator on $L^2(\mathbb{R}^n)$.
Remark: $K \in L_x^\infty L_y^1$ means $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| \mathrm{d}y < +\infty$.
(2) Suppose that $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Show that T is a compact operator on $L^2(\mathbb{R}^n)$.
(3) Suppose that K is compactly supported, and satisfies $|K(x, y)| \leq A|x - y|^{-n+\alpha}$ for some $\alpha > 0$, whenever $x, y \in \mathbb{R}^n$. Show that K is not necessarily $\in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, but T is still a compact operator on $L^2(\mathbb{R}^n)$.

Problem 3. Consider the Cauchy problem for the linear homogeneous wave equation in $\mathbf{R}^3 \times \mathbf{R}$:

$$\Box\,\phi = 0, \quad \phi(\mathbf{x}, \mathbf{0}) = \varphi(\mathbf{x}), \quad \partial_{\mathbf{t}}\phi(\mathbf{x}, \mathbf{0}) = \psi(\mathbf{x}).$$

Suppose that the smooth functions $\varphi(\mathbf{x}), \psi(\mathbf{x})$ have compact support and they only depend on the radial variable r, i.e. $\varphi(\mathbf{x}) = \varphi(\mathbf{r}), \psi(\mathbf{x}) = \psi(\mathbf{r})$.

- The solution ϕ to the above Cauchy problem only depends on the radial variable r and the time variable t, i.e. $\phi(\mathbf{x}, \mathbf{t}) = \phi(\mathbf{r}, \mathbf{t})$.

- Prove that for sufficiently large $T_0 > 0$, we have

$$\partial_r\left(r\phi\right)(0, t) \equiv 0, \quad \text{for all} \quad t \geq T_0.$$

- Let $u := t - r$, $\bar{u} := t + r$. Therefore ϕ can be viewed as a function of $(\bar{u}, u)$. Prove that if there is a $r_0 > 0$ such that

$$-\psi(r_0) + \partial_r\varphi(r_0) + \frac{1}{r_0}\varphi(r_0) \neq 0,$$

then there is a $u_0 \in \mathbf{R}$ such that

$$\lim_{\bar{u}\to\infty} \partial_u\left(r\phi\right)(\bar{u}, u_0) \neq 0.$$